§2.2 Probability: Axioms, Interpretation, Properties

Idea: Probability of an event is a fraction

$$P(\text{Event}) = \frac{\#\text{ times Event outcome occurs}}{\#\text{ times Experiment is run}}$$

↳ $P(\text{Event})$ gives likelihood of the Event occuring

- $P=0 \Rightarrow$ Event (basically) never occurs
- $P=1 \Rightarrow$ Event (basically) always occurs

("(basically)" is due to issues with infinite sample spaces)

Def: A probability measure is a function assigning a number to every event possible for an experiment

$$P(E) = \text{number}$$

satisfying a few basic properties.

Properties:

- Probability is always between 0 & 1

$$0 \le P(E) \le 1$$

- Probability of $\emptyset$ ("nothing") is 0 (Empty event has prob. 0)
  S ("anything") is 1 (Complete event has prob. 1)

$$P(\emptyset) = 0 \qquad P(S) = 1$$

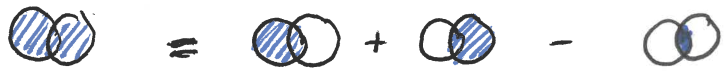
↑ Entire sample space

- If A is contained in B $(A \subseteq B)$
  then $P(A) \le P(B)$
  (Larger event $\Rightarrow$ Larger probability)

- If events E_i are all disjoint
  then $P(E_1 \cup E_2 \cup E_3 \cup \cdots) = P(E_1) + P(E_2) + \cdots$

- If A & B are any events (maybe not disjoint)
  then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



[For more than two events (i.e. A, B, C) this generalizes...]

- Complement probabilities add to 1

$$P(E') = 1 - P(E)$$

Probabilities are easiest to compute when the sample space is made of outcomes which are all equally likely ("equiprobable")

- If S is a set of N equiprobable outcomes then
  - $P(\{\text{outcome}\}) = \frac{1}{N}$
  - $P(E) = \frac{\#\text{ outcomes in } E}{N} = \frac{\#(E)}{\#(S)}$

Example: Roll one die.

$$P(\{\text{Result is} \geq 3\}) = \frac{\#\{\text{Rolls} \geq 3\}}{\#\{\text{Rolls}\}} = \frac{4}{6} = \frac{2}{3}$$

Example: Roll two dice

$$P\left(\left\{\begin{matrix}\text{sum is} \\ \geq 11\end{matrix}\right\}\right) = \frac{\#\{\text{Rolls with sum} \geq 11\}}{\#\{\text{Rolls}\}} = \frac{3}{36} = \frac{1}{12}$$

Note that our sample space here is all possible pairs of rolls — this is equiprobable because rolling $(D1 = 1 \,\&\, D2 = 3)$ is just as likely as rolling $(D1 = 4 \,\&\, D2 = 5)$ for example.

If our sample space were all possible sums of rolls, then it would not be equiprobable — rolling a sum of 12 is much less likely than 7.

Set operations are often useful when computing probabilities.

Example: Roll two dice

$$P\left(\left\{\begin{matrix}\text{sum is} \\ \leq 10\end{matrix}\right\}\right) = P\left(\left\{\begin{matrix}\text{sum is} \\ \geq 11\end{matrix}\right\}'\right)$$
$$= 1 - P\left(\left\{\begin{matrix}\text{sum is} \\ \geq 11\end{matrix}\right\}\right)$$
$$= 1 - \frac{3}{36} = 11/12$$

[Often it is faster to count the number of ways that something doesn't happen instead of the number of ways that it does]

Example: Roll two dice

$$P\left(\left\{\begin{matrix}\text{one of dice} \\ \text{is} \geq 5\end{matrix}\right\}\right) = P\left(\left\{\begin{matrix}D1 \text{ is} \\ \geq 5\end{matrix}\right\} \cup \left\{\begin{matrix}D2 \text{ is} \\ \geq 5\end{matrix}\right\}\right)$$
$$= P\left(\left\{\begin{matrix}D1 \text{ is} \\ \geq 5\end{matrix}\right\}\right) + P\left(\left\{\begin{matrix}D2 \text{ is} \\ \geq 5\end{matrix}\right\}\right) - P\left(\left\{\begin{matrix}D1 \,\&\, D2 \text{ are} \\ \text{both} \geq 5\end{matrix}\right\}\right)$$
$$= 2/6 + 2/6 - 4/36$$
$$= 20/36$$
$$= 5/9$$